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Umbrellas and Polytopal Approximation of the Euclidean Ball

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There are two positive, absolute constants c_1 and c_2 so that the volume of the difference set of the d-dimensional Euclidean ball B_2^d and an inscribed polytope with n vertices is larger than

$$c_1\, d \operatorname{vol}_d(B_2^d)\, n^{-2/(d-1)}$$

for $n \geqslant (c_2\, d)^{(d-1)/2}$. © 1997 Academic Press

We study here the approximation of a convex body in $\mathbb{R}^d$ by a polytope with at most n vertices. There are many means to measure the approximation, the two most common are the Hausdorff distance or the symmetric

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difference metric. The Hausdorff distance between two convex bodies K and C is

$$\mathrm{d}_H(K, C) = \max\{\max_{x \in C} \min_{y \in K} \|x - y\|, \max_{y \in K} \min_{x \in C} \|x - y\|\}$$

where $\|x\|$ is the Euclidean norm of x. The symmetric difference metric is the volume of the difference set.

$$\mathrm{d}_S(K, C) = \mathrm{vol}_d(K \triangle C).$$

Bronshtein and Ivanov [1] and Dudley [1, 2] showed that for every convex body K in $\mathbb{R}^d$ there is a constant $c = c(K, d)$ such that for every n there is a polytope P_n with at most n vertices and

$$\mathrm{d}_H(K, P_n) \leqslant cn^{-2/(d-1)}.$$

This can be used to show the same estimate for the symmetric difference metric. Gruber and Kenderov [9] showed that the inverse inequality holds if K has a C^2-boundary:

$$\mathrm{d}_S(K, P_n) \geqslant cn^{-2/(d-1)}.$$

Macbeath [10] showed that the approximation of a convex body is always better than that of the Euclidean ball. Gruber [8] obtained an asymptotic formula. If a convex body K in $\mathbb{R}^d$ has a C^2-boundary with everywhere positive curvature, then we have

$$\inf\{\mathrm{d}_S(K, P_n) \mid P_n \subset K \text{ and } P_n \text{ has at most n vertices}\}$$
$$\sim \frac{1}{2} \mathrm{del}_{d-1} \int_{\partial K} \kappa(x)^{1/(d+1)}\, d\mu(x) \left(\frac{1}{n}\right)^{2/(d-1)}$$

where del_{d-1} is a constant that is connected with triangulations. In [5] and [6], it was shown constructively that for all dimensions d, all convex bodies K, and all $n \geqslant 2$ there is a polytope P_n with n vertices that is contained in K such that

$$\mathrm{vol}_d(K) - \mathrm{vol}_d(P_n) \leqslant c_1\, d\, \mathrm{vol}_d(K)\, n^{-2/(d-1)}$$

where c_1 is a numerical constant. This estimate can also be derived from [1] and [2, 3]. So the question was whether the factor d was necessary, or, in other words, what is the order of magnitude of the constant del_d. The result in this paper shows that there are absolute positive constants c_1 and c_2 with

$$c_1 \leqslant \mathrm{del}_d \leqslant c_2.$$

In fact, we have

$$\mathrm{del}_{d-1} \leqslant \frac{32}{7} \left(\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})} \right)^{2/(d-1)}$$

this follows from estimate (1) below.

In this paper we want to show that there is a universal constant $c > 0$ such that the volume of the difference set of the d-dimensional Euclidean ball and an inscribed polytope with n vertices is larger than

$$c\, d \,\mathrm{vol}_d(B_2^d)\, n^{-2/(d-1)}.$$

We want to reduce the computation of the volume of the difference set to that of the following set: The set between a $d-1$-dimensional face of the polytope and the boundary of the sphere. Intuitively it is clear that the faces should be simplices and that the polytope should have rather regular features. This leads us to the assumption that the volume of the set between a $d-1$-dimensional face of the polytope and the boundary of the sphere equals in average approximately the surface area of the face times the height of the cap of the Euclidean ball that is determined by that face.

There are two technical difficulties. The number of faces does not necessarily correspond to the number of vertices. In fact, a heuristic argument shows that the number of faces is of the order of the number of vertices times $d^{d/2}$. Secondly, although we may assume that the faces are simplices, we may not assume that they are regular or close to regular. This is expressed in the following way. If F is a face and H the hyperplane containing F then the distance of the centers of gravity of F and $H \cap B_2^d$ may be large.

Hyperplanes are usually denoted by H and the closed halfspaces associated with H by H^+ and H^-. $H(x, \xi)$ is the hyperplane that passes through x and is orthogonal to ξ.

The $d-1$-dimensional faces of a polytope in $\mathbb{R}^d$ are denoted by F_j. The hyperplanes containing F_j are denoted by H_j. H_j^+ denotes the halfspace containing P.

For a polytope P that is contained in B_2^d the height or width of $B_2^d \cap H_j^-$ is h_j and the radius of $B_2^d \cap H_j$ is r_j.

$\mathbf{cg}(M)$ is the center of gravity of the set M.

$[A, B]$ denotes the convex hull of the sets A and B. The radial projection $\mathbf{rp}(M)$ of a set M in B_2^d is

$$\mathbf{rp}(M) = \{\xi \in \partial B_2^d \,|\, [0, \xi] \cap M \neq \varnothing\}.$$

THEOREM 1. *There are two positive constants c_1 and c_2 so that we have for all d, $d \geqslant 2$, and all n, $n \geqslant (c_2 d)^{(d-1)/2}$, and all polytopes P_n that are contained in the Euclidean unit ball B_2^d and have n vertices*

$$vol_d(B_2^d) - vol_d(P_n) \geqslant c_1\, d\, vol_d(B_2^d)\, n^{-2/(d-1)}.$$

In particular we have by Theorem 1 that

$$\mathrm{vol}_d(B_2^d) - \mathrm{vol}_d(P_n) \geqslant \frac{c_1}{c_2} \mathrm{vol}_d(B_2^d)$$

if $n \leqslant (c_2 d)^{(d-1)/2}$.

LEMMA 2. (i) *For all x, $0 < x$, there is a θ, $0 < \theta < 1$, such that*

$$\Gamma(x+1) = \sqrt{2\pi}\, x^{x+1/2} \exp\left(-x + \frac{\theta}{12x}\right).$$

(ii) $$vol_d(B_2^d) = \frac{\pi^{d/2}}{\Gamma((d/2)+1)} \leqslant \frac{\pi^{(d-1)/2}(2e)^{d/2}}{d^{(d+1)/2}}.$$

The following lemma is due to Bronshtein and Ivanov [1] and Dudley [2, 3].

LEMMA 3. *For all dimensions d, $d \geqslant 2$, and all natural numbers n, $n \geqslant 2d$, there is a polytope Q_n that has n vertices and is contained in the Euclidean ball B_2^d such that*

$$d_H(Q_n, B_2^d) \leqslant \frac{16}{7} \left(\frac{vol_{d-1}(\partial B_2^d)}{vol_{d-1}(B_2^{d-1})} \right)^{2/(d-1)} n^{-2/(d-1)}.$$

In particular, since a Q_n which satisfies the hypothesis of Lemma 3 contains the Euclidean ball of radius $1 - \mathrm{d}_H(Q_n, B_2^d)$, it follows that

$$\begin{aligned} \mathrm{d}_S(Q_n, B_2^d) &\leqslant \mathrm{vol}_d(B_2^d)\{1 - (1 - \mathrm{d}_H(Q_n, B_2^d))^d\} \\ &\leqslant \mathrm{vol}_d(B_2^d) \left\{ 1 - \left(1 - \frac{16}{7} \left(\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})} \right)^{2/(d-1)} n^{-2/(d-1)} \right)^d \right\} \end{aligned} \tag{1}$$

and

$$\left\{ 1 - \frac{16}{7} \left(\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})} \right)^{2/(d-1)} n^{-2/(d-1)} \right\}^{d-1} \mathrm{vol}_{d-1}(\partial B_2^d) \leqslant \mathrm{vol}_{d-1}(\partial Q_n). \tag{2}$$

We have that

$$\begin{aligned}\mathrm{vol}_{d-1}(\partial B_2^d) = d\,\mathrm{vol}_d(B_2^d) &= d\,\frac{\pi^{d/2}}{\Gamma((d/2)+1)}\\ &= d\sqrt{\pi}\,\frac{\Gamma((d-1)/2+1)}{\Gamma((d/2)+1)}\,\mathrm{vol}_{d-1}(B_2^{d-1})\\ &\leqslant d\sqrt{\pi}\,\mathrm{vol}_{d-1}(B_2^{d-1}).\end{aligned}$$

Since $d^{2/(d-1)} \leqslant 4$ and $(1-t)^d \geqslant 1-dt$ we get from (1)

$$\begin{aligned}\mathrm{d}_S(Q_n, B_2^d) &\leqslant (1-(1-\tfrac{64}{7}\pi n^{-2/(d-1)})^d)\,\mathrm{vol}_d(B_2^d)\\ &\leqslant \tfrac{64}{7}\,\pi\, dn^{-2/(d-1)}\,\mathrm{vol}_d(B_2^d). \qquad (3)\end{aligned}$$

Similarly we get from (2) that we have for $n \geqslant (\frac{128}{7}\pi\, d)^{(d-1)/2}$.

$$\mathrm{vol}_{d-1}(\partial B_2^d) \leqslant 2\,\mathrm{vol}_{d-1}(\partial Q_n). \qquad (4)$$

For the sake of completeness we include the proof of Lemma 3. The arguments are from [1].

Proof. For every n there is a $\theta_n > 0$ and a set $\{x_1, ..., x_n\} \subset \partial B_2^d$ so that for all $i \neq j$ we have

$$\|x_i - x_j\| \geqslant \theta_n$$

and so that for every $x \in \partial B_2^d$ there is i such that

$$\|x - x_i\| \leqslant \theta_n.$$

We choose Q_n to be the convex hull of $\{x_1, ..., x_n\}$. We have

$$\mathrm{d}_H(Q_n, B_2^d) \leqslant \tfrac{1}{2}\theta_n^2.$$

If not, then there is $x \in \partial B_2^d$ such that the Euclidean ball with radius $\frac{1}{2}\theta_n^2$ and center x and Q_n have an empty intersection. By the theorem of Hahn–Banach there is a hyperplane separating Q_n and $B_2^d(x, \frac{1}{2}\theta_n^2)$. This hyperplane cuts off a cap of height greater than $\frac{1}{2}\theta_n^2$. The point at the top of this cap has a distance greater than θ_n from all x_i, $i = 1, ..., n$. This cannot be.

Now we estimate θ_n from above. The caps

$$\partial B_2^d \cap H^-((1-\tfrac{1}{8}\,\theta_n^2)\,x_i, x_i) \qquad i = 1, ..., n$$

have disjoint interiors. Therefore we get

$$\begin{aligned}\mathrm{vol}_{d-1}(\partial B_2^d) &\geqslant \sum_{i=1}^{n} \mathrm{vol}_{d-1}\left(\partial B_2^d \cap H^-((1-\tfrac{1}{8}\theta_n^2)\, x_i, x_i)\right)\\ &\geqslant n\left(\tfrac{1}{2}\theta_n \sqrt{1-\tfrac{1}{16}\theta_n^2}\right)^{d-1} \mathrm{vol}_{d-1}(B_2^{d-1}).\end{aligned}$$

We obtain

$$\frac{1}{2}\,\theta_n \sqrt{1-\frac{1}{16}\,\theta_n^2} \leqslant \left(\frac{1}{n}\,\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{1/(d-1)}$$

For $n=2d$ we get that $\theta_n \leqslant \sqrt{2}$. Indeed, just consider the set $\{e_1, ..., e_d, -e_1, ..., -e_d\}$. Thus it follows

$$\frac{\theta_n}{2}\sqrt{\frac{7}{8}} \leqslant \left(\frac{1}{n}\,\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{1/(d-1)}$$

and thus

$$\frac{1}{2}\,\theta_n^2 \leqslant \frac{16}{7}\left(\frac{1}{n}\,\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{2/(d-1)}. \quad \blacksquare$$

LEMMA 4. (i) *For $k=0, 1, 2, \ldots$ and $d=1, 2, \ldots$ we have*

$$\int_{\mathbb{R}_+^d} \left(\sum_{i=1}^{d} y_i\right)^k \exp\left(-\left(\sum_{i=1}^{d} y_i\right)^2\right) dy = \frac{\Gamma((k+1)/2)}{2(d-1)!}.$$

(ii) $$\int_{\mathbb{R}_+^d} \left(\sum_{i=1}^{d} y_i^2\right) \exp\left(-\left(\sum_{i=1}^{d} y_i\right)^2\right) dy = \frac{d^2}{2(d+1)!}\,\Gamma\left(\frac{d}{2}\right).$$

(iii) *For $i \neq j$ we have*

$$\int_{\mathbb{R}_+^d} y_i y_j \exp\left(-\left(\sum_{i=1}^{d} y_i\right)^2\right) dy = \frac{\Gamma(d/2)}{4(d+1)(d-1)!}.$$

Proof. This is an easy consequence of Fubini theorem with the change of variables $x_d = \sum_{i=1}^{d} y_i$ and $x_i = y_i$, $i = 1, ..., d-1$. $\blacksquare$

For the following lemma compare also [12].

LEMMA 5. *Let $x_1, ..., x_d$ be points on the Euclidean sphere of radius* 1, *S the simplex $[x_1, ..., x_d]$, and* $\mathbf{rp}(S)$ *the radial projection of S, i.e., the* *spherical simplex of the points $x_1, ..., x_d$. Let X be the matrix whose columns are the vectors $x_1, ..., x_d$. Then we have*

$$vol_{d-1}(\mathbf{rp}(S)) = \frac{2}{\Gamma(d/2)} |\det(X)| \int_{\mathbb{R}^d_+} \exp(-y^t X^t X y)\, dy$$

and

$$vol_d([0, \mathbf{rp}(S)]) = \frac{1}{\Gamma((d/2)+1)} |\det(X)| \int_{\mathbb{R}^d_+} \exp(-y^t X^t X y)\, dy.$$

Proof. We have

$$\mathrm{vol}_d(B_2^d) = \frac{\pi^{d/2}}{\Gamma((d/2)+1)}$$

and

$$\int_{\mathbb{R}^d} e^{-\|z\|^2}\, dz = \pi^{d/2}.$$

Therefore we get

$$\mathrm{vol}_d([0, \mathbf{rp}(S)]) = \frac{1}{\Gamma((d/2)+1)} \int_{\{z = t\xi \mid \xi \in S \text{ and } t \in \mathbb{R}_+\}} e^{-\|z\|^2}\, dz.$$

Using the substitution $z = Xy$ we get the latter expression equals

$$\frac{1}{\Gamma((d/2)+1)} |\det(X)| \int_{\mathbb{R}^d_+} e^{-y^t X^t X y}\, dy. \quad \blacksquare$$

LEMMA 6. *Let $x_1, ..., x_d$ be points on the Euclidean sphere of radius 1, S the simplex $[x_1, ..., x_d]$, and let $\mathbf{rp}(S)$ be the radial projection of the simplex S. Let H be the hyperplane containing the simplex $[x_1, ..., x_d]$ and r the radius of the $d-1$-dimensional Euclidean ball $H \cap B_2^d$. Then we have*

$$vol_d([0, \mathbf{rp}(S)]) - vol_d([0, S]) \geqslant \frac{d^2}{2(d+1)} \left(1 - \left\| \frac{1}{d} \sum_{i=1}^{d} x_i \right\|^2\right) vol_d([0, S])$$

and

$$vol_d([0, \mathbf{rp}(S)]) - vol_d([0, S]) \geqslant \frac{d\sqrt{1-r^2}}{2(d+1)} \left(1 - \left\| \frac{1}{d} \sum_{i=1}^{d} x_i \right\|^2\right) vol_{d-1}(S).$$

Proof. By Lemma 5 we have

$$\begin{aligned}
&\mathrm{vol}_d([0, \mathbf{rp}(S)]) - \mathrm{vol}_d([0, S]) \\
&\quad = \frac{1}{\Gamma((d/2)+1)} |\det(X)| \int_{\mathbb{R}^d_+} \exp(-y^t X^t X y)\, dy - \frac{|\det(X)|}{d!}.
\end{aligned}$$

By Lemma 4(i) with $k=0$ the last expression equals

$$\begin{aligned}
&\frac{1}{\Gamma((d/2)+1)} |\det(X)| \int_{\mathbb{R}^d_+} \left\{\exp(-y^t X^t X y) - \exp\left(-\left(\sum_{i=1}^d y_i\right)^2\right)\right\} dy \\
&\quad = \frac{1}{\Gamma((d/2)+1)} |\det(X)| \\
&\qquad \times \int_{\mathbb{R}^d_+} \left\{\exp\left(\left(\sum_{i=1}^d y_i\right)^2 - y^t X^t X y\right) - 1\right\} \exp\left(-\left(\sum_{i=1}^d y_i\right)^2\right) dy.
\end{aligned}$$

We use now the inequality $1+t \leqslant e^t$ and get that the above expression is greater than or equal to

$$\begin{aligned}
&\frac{1}{\Gamma((d/2)+1)} |\det(X)| \int_{\mathbb{R}^d_+} \left\{\left(\sum_{i=1}^d y_i\right)^2 - y^t X^t X y\right\} \exp\left(-\left(\sum_{i=1}^d y_i\right)^2\right) dy \\
&\quad = \frac{1}{\Gamma((d/2)+1)} |\det(X)| \sum_{i,j=1}^d (1 - \langle x_i, x_j\rangle) \int_{\mathbb{R}^d_+} y_i y_j \exp(-(y^t y)^2)\, dy.
\end{aligned}$$

Since we have $\langle x_i, x_i\rangle = 1$ for $i = 1, ..., d$, we get by Lemma 4(iii) for the above expression

$$\begin{aligned}
&= \frac{1}{\Gamma((d/2)+1)} |\det(X)| \sum_{i,j=1}^d (1 - \langle x_i, x_j\rangle) \frac{\Gamma(d/2)}{4(d+1)(d-1)!} \\
&= \frac{1}{2(d+1)!} |\det(X)| \left(d^2 - \left\|\sum_{i=1}^d x_i\right\|^2\right) \\
&= \frac{d\sqrt{1-r^2}}{2(d+1)} \left(1 - \left\|\frac{1}{d}\sum_{i=1}^d x_i\right\|^2\right) \mathrm{vol}_{d-1}(S). \quad \blacksquare
\end{aligned}$$

LEMMA 7. *Let A be a measurable subset of B_2^d such that the center of gravity of A is contained in a cap of height Δ, $\Delta \leqslant 1$. Then there is a cap C of height 2Δ so that*

$$2\, vol_d(C \cap A) \geqslant vol_d(A).$$

Proof. Let us consider B_2^d with the origin situated at the "south pole", i.e. $B_2^d = \{x = (\xi_1, ..., \xi_d) \mid \sum_{i=1}^{d-1} \xi_i^2 + (\xi_d - 1)^2 \leqslant 1\}$. W.l.g. assume that the center of gravity of the set A is on the ξ_d axis at the point $(0, ..., 0, \Delta)$ where $0 < \Delta \leqslant 1$. Let $a(t)$ be the $d-1$-dimensional Lebesgue measure of the intersection of A with the hyperplane $\{x \mid \xi_d = t\}$. Then $\mathrm{vol}_d(A) = \int_0^2 a(t)\, dt$ and $\Delta = (1/\mathrm{vol}_d(A)) \int_0^2 t a(t)\, dt$. Let C be the cap $B_2^d \cap \{x \mid \xi_d \leqslant 2\Delta\}$. We obtain

$$
\begin{aligned}
2\mathrm{vol}_d(C \cap A) &= 2 \int_{t < 2\Delta} a(t)\, dt \\
&= 2\mathrm{vol}_d(A) - 2 \int_{t \geqslant 2\Delta} a(t)\, dt \\
&\geqslant 2\mathrm{vol}_d(A) - \int_{t \geqslant 2\Delta} \frac{t}{\Delta} a(t)\, dt - \int_{t < 2\Delta} \frac{t}{\Delta} a(t)\, dt \\
&= \mathrm{vol}_d(A). \quad \blacksquare
\end{aligned}
$$

LEMMA 8. *Let P_n be a simplicial polytope with vertices $x_1, ..., x_n$ that are elements of ∂B_2^d. Let F_j, $j = 1, ..., m$ be the $d-1$-dimensional faces of P_n, H_j the hyperplane containing F_j, h_j the height of the cap $B_2^d \cap H_j^-$, and r_j the radius of $B_2^d \cap H_j$. Let $\mathcal{N}$ be the set of integers j so that*

$$
h_j \leqslant \frac{1}{8} \left(\frac{vol_{d-1}(\partial P_n)}{vol_{d-1}(\partial B_2^d)} \frac{1}{4n} \right)^{2/(d-1)}.
$$

Then we have

$$
vol_{d-1}\left(\bigcup_{j \in \mathcal{N}} F_j \right) \leqslant \frac{1}{4} vol_{d-1}(\partial P_n).
$$

Proof. We put

$$
\mathcal{N}_i = \{ j \in \mathcal{N} \mid x_i \in F_j \} \qquad i = 1, ..., n
$$

and

$$
\rho = \frac{1}{8} \left(\frac{\mathrm{vol}_{d-1}(\partial P_n)}{\mathrm{vol}_{d-1}(\partial B_2^d)} \frac{1}{4n} \right)^{2/(d-1)}.
$$

Since $h_j \leqslant \rho$ we have that $\bigcup_{j \in \mathcal{N}_i} F_j$ is contained in $B_2^d(x_i, 2\sqrt{2\rho})$. $\bigcup_{j \in \mathcal{N}_i} F_j$ is a subset of the boundary of the convex set $P_n \cap B_2^d(x_i, 2\sqrt{2\rho})$. Thus we get

$$\mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathcal{N}_i} F_j\right) \leqslant \mathrm{vol}_{d-1}(\partial(P_n \cap B_2^d(x_i, 2\sqrt{2\rho}))).$$

Since $P_n \cap B_2^d(x_i, 2\sqrt{2\rho})$ is a convex subset of the convex set $B_2^d(x_i, 2\sqrt{2\rho})$ we get

$$\mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathcal{N}_i} F_j\right) \leqslant (8\rho)^{(d-1)/2} \,\mathrm{vol}_{d-1}(\partial B_2^d) \leqslant \frac{1}{4n} \mathrm{vol}_{d-1}(\partial P_n).$$

Therefore we get

$$\begin{aligned}\mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathcal{N}} F_j\right) &= \mathrm{vol}_{d-1}\left(\bigcup_{i=1}^{n} \bigcup_{j \in \mathcal{N}_i} F_j\right) \\ &\leqslant \sum_{i=1}^{n} \mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathcal{N}_i} F_j\right) \leqslant \frac{1}{4} \mathrm{vol}_{d-1}(\partial P_n). \quad \blacksquare\end{aligned}$$

LEMMA 9. *Let P_n be a simplicial polytope with vertices $x_1, ..., x_n$ that are elements of ∂B_2^d. Let F_j, $j = 1, ..., m$ be the $d-1$-dimensional faces of P_n, H_j the hyperplane containing F_j, h_j the height of the cap $B_2^d \cap H_j^-$, and r_j the radius of $B_2^d \cap H_j$. Assume that we have for all j, $j = 1, ..., m$*

$$h_j \leqslant \frac{16}{7}\left(2\,\frac{vol_{d-1}(\partial B_2^d)}{vol_{d-1}(B_2^{d-1})}\right)^{2/(d-1)} n^{-2/(d-1)}$$

and assume that

$$vol_{d-1}(\partial B_2^d) \leqslant 2\, vol_{d-1}(\partial P_n).$$

Let $\mathcal{M}$ be the set of integers j so that

$$\|\mathbf{cg}(F_j) - \mathbf{cg}(H_j \cap B_2^d)\| \geqslant \frac{2^{22}-1}{2^{22}} r_j.$$

Then we have

$$vol_{d-1}\left(\bigcup_{j \in \mathcal{M}} F_j\right) \leqslant \frac{1}{4} vol_{d-1}(\partial P_n).$$

Proof. We put

$$\theta = \frac{16}{7}\left(2\,\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{2/(d-1)} n^{-2/(d-1)}$$

$$\leqslant \frac{16}{7}\,(2d\sqrt{\pi})^{2/(d-1)}\, n^{-2/(d-1)}.$$

Since $k_j \leqslant \theta$ we have for all j, $j=1, ..., m$

$$r_j \leqslant \sqrt{2\theta}.$$

We have that $\mathbf{cg}(F_j)$ is contained in a cap of height $2^{-22} r_j$ of the $d-1$-dimensional Euclidean ball $H_j \cap B_2^d$. By Lemma 7 there is a subset $\tilde{F}_j$ of F_j so that $\tilde{F}_j$ is contained in a cap of height $2^{-21} r_j$ and

$$\mathrm{vol}_{d-1}(F_j) \leqslant 2\,\mathrm{vol}_{d-1}(\tilde{F}_j).$$

Thus the diameter of $\tilde{F}_j$ is less than $2^{-9} r_j \leqslant \sqrt{2\theta}/512$. The set of all integers j such that $x_i \in \tilde{F}_j$ is denoted by $\mathscr{M}_i$. We have that $\bigcup_{j \in \mathscr{M}_i} \tilde{F}_j$ is a subset of the boundary of the convex set $P_n \cap B_2^d(x_i, 2^{-9}\sqrt{2\theta})$ and has a smaller surface area than $B_2^d(x_i, 2^{-9}\sqrt{2\theta})$.

$$\mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathscr{M}_i} \tilde{F}_j\right) \leqslant \left(\frac{\sqrt{2\theta}}{512}\right)^{d-1} \mathrm{vol}_{d-1}(\partial B_2^d)$$

$$\leqslant \frac{4d\sqrt{\pi}}{n}\left(\frac{\sqrt{32}}{512\sqrt{7}}\right)^{d-1} \mathrm{vol}_{d-1}(\partial P_n).$$

Since $d \leqslant 2^{d-1}$ we get that the latter expression is smaller than

$$\frac{4\sqrt{\pi}}{n}\left(\frac{\sqrt{2}}{128}\right)^{d-1} \mathrm{vol}_{d-1}(\partial P_n) \leqslant \frac{\sqrt{2\pi}}{32n}\,\mathrm{vol}_{d-1}(\partial P_n) \leqslant \frac{1}{8n}\,\mathrm{vol}_{d-1}(\partial P_n).$$

Therefore we get

$$\mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathscr{M}} F_n\right) = \mathrm{vol}_{d-1}\left(\bigcup_{i=1}^{n} \bigcup_{j \in \mathscr{M}_i} F_j\right) \leqslant \sum_{i=1}^{n} \mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathscr{M}_i} F_j\right)$$

$$\leqslant 2 \sum_{i=1}^{n} \mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathscr{M}_i} \tilde{F}_j\right) \leqslant \frac{1}{4}\,\mathrm{vol}_{d-1}(\partial P_n). \quad \blacksquare$$

Proof of Theorem 1. We consider numbers of vertices n such that $n \geqslant (\frac{512}{7}\pi d)^{(d-1)/2}$. Let P_n be a polytope with n vertices so that $\mathrm{vol}_d(B_2^d) - \mathrm{vol}_d(P_n)$ is minimal. Let Q_n be a polytope with n vertices so that $\mathrm{d}_H(B_2^d, Q_n)$ is minimal. By Lemma 3 we have that for all j

$$\mathrm{d}_H(B_2^d, Q_n) \leqslant \frac{16}{7}\left(\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{2/(d-1)} n^{-2/(d-1)}.$$

We consider now the convex hull of P_n and Q_n.

$$P = [P_n, Q_n].$$

P has at most $2n$ vertices. Its $d-1$-dimensional faces are denoted by F_j, $j = 1, ..., m$. H_j is the hyperplane containing F_j, h_j the height of the cap $B_2^d \cap H_j^-$, and r_j the radius of $B_2^d \cap H_j$. We may assume that P is simplicial. We have that

$$h_j \leqslant \mathrm{d}_H(B_2^d, Q_n) \leqslant \frac{16}{7}\left(\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{2/(d-1)} n^{-2/(d-1)}.$$

By the assumption on n we have that

$$h_j \leqslant \tfrac{1}{8} \qquad \text{and} \qquad r_j = \sqrt{2h_j - h_j^2} \leqslant \tfrac{1}{2}. \tag{5}$$

Also we have by (4) that

$$\mathrm{vol}_{d-1}(\partial B_2^d) \leqslant 2\,\mathrm{vol}_{d-1}(\partial Q_n) \leqslant 2\,\mathrm{vol}_{d-1}(\partial P).$$

We apply Lemmas 8 and 9 to P that has at most $2n$ vertices. Thus a factor 2 enters the estimates. Let $\mathscr{L}$ be the set of integers j so that

$$\frac{1}{8}\left(\frac{\mathrm{vol}_{d-1}(\partial P_n)}{\mathrm{vol}_{d-1}(\partial B_2^d)}\frac{1}{8n}\right)^{2/(d-1)} \leqslant h_j \leqslant \frac{16}{7}\left(\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{-1}(B_2^{d-1})}\frac{1}{n}\right)^{2/(d-1)} \tag{6}$$

and

$$\|\mathbf{cg}(F_j) - \mathbf{cg}(H_j \cap B_2^d)\| < \frac{2^{22}-1}{2^{22}}\, r_j. \tag{7}$$

We have

$$\mathrm{vol}_{d-1}\left(\bigcup_{j \in \mathscr{L}} F_j\right) \geqslant \frac{1}{2}\,\mathrm{vol}_{d-1}(\partial P). \tag{8}$$

We apply Lemma 6

$$\begin{aligned}\mathrm{vol}_d(B_2^d)-\mathrm{vol}_d(P_n)&\geqslant \mathrm{vol}_d(B_2^d)-\mathrm{vol}_d(P)\\&\geqslant \sum_{j\in\mathscr{L}}(\mathrm{vol}_d([0,\mathbf{rp}(F_j)])-\mathrm{vol}_d([0,F_j]))\\&\geqslant \sum_{j\in\mathscr{L}}\frac{\sqrt{1-r_j^2}}{4}(1-\|\mathbf{cg}(F_j)\|^2)\,\mathrm{vol}_{d-1}(F_j).\end{aligned}$$

By (5) we have $r_j\leqslant\frac{1}{2}$ and get that the latter expression is greater than

$$\sum_{j\in\mathscr{L}}\frac{1}{8}(1-\|\mathbf{cg}(F_j)\|^2)\,\mathrm{vol}_{d-1}(F_j).$$

We have

$$\|\mathbf{cg}(F_j)\|^2=(1-h_j)^2+\|\mathbf{cg}(F_j)-\mathbf{cg}(H_j\cap B_2^d)\|^2.$$

By (7) we get for $j\in\mathscr{L}$

$$\begin{aligned}1-\|\mathbf{cg}(F_j)\|^2&\geqslant 1-(1-h_j)^2-\left(\frac{2^{22}-1}{2^{22}}r_j\right)^2\\&=1-(1-h_j)^2-\left(\frac{2^{22}-1}{2^{22}}\right)^2(2h_j-h_j^2)\\&=(2^{-21}-2^{-44})(2h_j-h_j^2)\geqslant 2^{-21}h_j.\end{aligned}$$

Therefore

$$\mathrm{vol}_d(B_2^d)-\mathrm{vol}(P)\geqslant\frac{1}{2^{24}}\sum_{j\in\mathscr{L}}h_j\,\mathrm{vol}_{d-1}(F_j).$$

By (6) we get that this expression is greater than

$$\frac{1}{2^{27}}\left(\frac{\mathrm{vol}_{d-1}(\partial P)}{\mathrm{vol}_{d-1}(\partial B_2^d)}\frac{1}{8n}\right)^{2/(d-1)}\sum_{j\in\mathscr{L}}\mathrm{vol}_{d-1}(F_j),$$

and by (8) this expression is greater than

$$\begin{aligned}&\frac{1}{2^{29}}\left(\frac{\mathrm{vol}_{d-1}(\partial P)}{\mathrm{vol}_{d-1}(\partial B_2^d)}\frac{1}{8n}\right)^{2/(d-1)}\mathrm{vol}_{d-1}(\partial P)\\&\geqslant\frac{1}{2^{36}}\mathrm{vol}_{d-1}(\partial B_2^d)\,n^{-2/(d-1)}=\frac{1}{2^{36}}d\,\mathrm{vol}_d(B_2^d)\,n^{-2/(d-1)}.\quad\blacksquare\end{aligned}$$

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